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Optimization of support positions to minimize the maximal deflection of structures

D. Wang *

Department of Aeronautical Structural Engineering, Northwestern Polytechnical University, P.O. Box 118#, Xi'an, Shaanxi 710072, PR China

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Abstract

In this paper, the design sensitivity analysis for the deflection of a beam or plate structure is first investigated with respect to the position of a simple support using the discrete method. Both elastic and rigid supports are taken into account, and closed-form formulae for the deflection sensitivity are developed straightforwardly. Then, on the basis of the design sensitivity analysis, a heuristic optimization algorithm, called the evolutionary shift method, is presented for support position optimization to minimize the maximal deflection of a structure with a fixed grid mesh scheme. In each iterative loop, the support with the highest efficiency is shifted in priority. To facilitate the convergence of the process, a polynomial interpolation technique is employed to evaluate the solution more accurately. The optimal solution is achieved gradually with a minimum modification of the support layout design. Finally, three numerical examples are presented to demonstrate the validities of the sensitivity analysis and the optimization method. Results show that support optimization can improve the structural behavior significantly.
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Keywords: Support position optimization; Sensitivity analysis; Maximal deflection minimization

1. Introduction

Design optimizations of structural topology, geometry (shape) and sizing with fixed support positions have been researched extensively over the last decades. A bulk of publications can be found in literature for a variety of optimization approaches. Recently, the optimization of support positions or support layout has been a very active topic for reducing the maximal deflection or bending moment (Imam and Al-Shihri, 1996), raising the fundamental frequency of a structural system (Won and Park, 1998), increasing the buckling load factor (Liu et al., 2000), etc. Much as known, supports are utilized to restrain the structure and prevent it from deflecting excessively. They play an important role in the structural design, and should be considered carefully over the structural behaviors. This is because a small amount of movement in

* Tel./fax: +86-29-8849-3386.
E-mail address: wangdng66@yahoo.com (D. Wang).

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support positions can influence the structural response significantly and, therefore, improve the quality of the structural design notably.

Support position optimization may arise in almost all structural design projects, especially in building constructions, workpiece machining, printed circuit boards, marine and aircraft structures. However, support position optimization for decreasing structural deflections has not yet been fully investigated. It still remains the most challenging task for researchers because the maximal deflection of a structure, as the objective function of the problem in this paper, is highly nonlinear and nonglossy with respect to support positions. Generally, the maximal deflection does not occur at a fixed point, i.e., it often switches its position from one point to another during the solution process, which often entails the algorithms much greater mathematical difficulties. Hence, relatively few publications are available so far.

Imam and Al-Shihri (1996) explained the main importance for optimal design of support positions in engineering practices. Instead of minimizing the mass by determining the optimal element sizes, they firstly studied the support position optimization with the objective of minimizing the maximal deflection and bending moment of a frame structure, respectively. They carried out the optimization by using the feasible direction method. Marcelin (2001) dealt with the support position optimization to minimize the strain or stress in the workpiece during the machining process with the genetic algorithms.

Design sensitivity analysis aims at studying the effect of design variable changes on the response of a structural system. It plays a vital role in iterative optimization approaches, especially, in gradient-based optimization methods, where accurate calculations of design sensitivities are desirable. With sensitivity information, design optimization can proceed consecutively without trial and error. Therefore, the sensitivity analysis has been an active research topic in the field of structural optimization. Commonly, two approaches are utilized to evaluate sensitivities of structural responses with respect to design variables. Imam and Al-Shihri (1996) used the finite difference approximation for design sensitivities of structural deflections. The finite difference technique is independent of the response functions and analysis types (e.g., static or dynamic analysis), which makes it very popular and simple for implementation since no prior knowledge of the response is required. However, an undeniable fact is that this technique is computationally burdensome and prohibitive time-consuming if the number of the design variables is large. Moreover, it often has accuracy problems because no acceptable step size could be found for a general problem (Haftka et al., 1990; Adelman and Haftka, 1993; Hsu, 1994).

The analytical method for sensitivity analysis is more efficient than the finite difference technique. It needs the prior knowledge regarding the characteristics of the structural behavior to develop closed-form expressions of the design sensitivity. Usually, there are two basic approaches: the differential approach and the variational approach. The differential approach is based on the direct differentiation of structural response by using direct or adjoint techniques (Haug et al., 1986), whereas the variational approach is based upon the principle of virtual work by differentiating the variational state equation of the structural system. Much as known, the calculation of the design sensitivity is often the major computational cost of an optimization process. It becomes, therefore, increasingly important to develop higher efficient algorithms for the sensitivity computation of deflection with regard to a support position. However, little work is available at present.

The problem under investigation in this paper is to optimize the positions of simple or point supports to minimize the maximal deflection of beam or plate structures. Both elastic and rigid supports are taken into account. Supports are assumed to hold the structure at the nodes of the finite element (FE) model and act only on the transverse displacements of the supported points. The main objective of this study is twofold. First, the sensitivity analysis of a deflection with respect to a support position is implemented by using the discrete method. According to the element shape functions of the FE analysis, the closed-form sensitivity formulations are developed neatly and straightforwardly. As the structural analysis resorts most regularly to the numerical execution with the FE method, such a derivation of the design sensitivity is consistent with the structural numerical analysis. Second, a heuristic optimization algorithm, called the evolutionary shift method, is presented for minimizing the maximal absolute deflection of a structure. Based on the sensitivity analysis, the most efficient support is identified and shifted along the elementary edges with the move step (interval) of the elementary size. Then, the solution can be obtained gradually with a minimum modification of the initial support layout design.

With the fixed grid mesh scheme, however, it is not always possible that the optimal support positions are at the FE nodes exactly. To overcome this difficulty and facilitate the convergence of the optimization process, usually, two techniques may be used to find the optimal position. An approach, which has been extensively used in structural optimizations, is to subdivide the elements in the local region near the optimal design. An alternative presented in this paper is to estimate the optimal support position by a polynomial interpolation using the sensitivity information at the FE nodes, which, in turn, makes the solution insensible to the FE model. Finally, three numerical examples will be given to illustrate the validity of the formulation for the design sensitivity and the effectiveness of the proposed optimization approach. Results show that support optimization can make a substantial improvement in the structural behavior, and deserves careful consideration in a practical design.

2. Problem formulation

Generally, supports are utilized to hold the structure firmly and prevent it from deflecting excessively. In support position optimization problems, the coordinates of simple support positions are referred to as design variables. These supports will be located within a prescribed domain to minimize the maximal absolute deflection of a structure. In addition, some supports may be linked so as to keep the structural system symmetric and/or limit the number of design variables. Therefore, the optimization problem of support positions can be defined mathematically as

$$\text{Minimize} \quad \max(|\delta_i|, \quad i = 1, \ldots, m) \tag{1}$$

$$\text{Subject to} \ \begin{cases} \underline{a}_j \leqslant a_j \leqslant \bar{a}_j \quad (j = 1, \ldots, n) \\ a_{\mathrm{d}} = f(a_j) \end{cases} \tag{2}$$

where $|\delta_i|$ is the absolute value of the ith nodal deflection of the structural FE model, and m is the total number of nodal deflections of interest. a_j indicates the design variable, representing the coordinate of the jth independent support position of n simple supports, and a_{d} is a dependent support coordinate. $\underline{a}_j$ and $\bar{a}_j$ denote the lower and upper bounds of the support positions, respectively.

Because the 'maximal' and 'absolute' values used in the objective function in Eq. (1) does not refer to the same point deflection during the design optimization, the maximal deflection may often change its position from one point to another. Consequently, it presents a practical barrier to an algorithm and frustrates its usefulness in the problem since abrupt changes in the derivatives of the objective function often occur in the solution process (Imam and Al-Shihri, 1996). In addition, more nodal deflections need to be taken into account in Eq. (1), which implies that m should take a larger number.

3. Deflection sensitivity with respect to support position

3.1. General case

First, the first-order derivative of a specific nodal deflection with regard to the change of a simple support position is derived. The force equilibrium equation of a loaded structure is

$$[K]\{u\} = \{P\} \tag{3}$$

where $[K]$ is the global stiffness matrix of the structural system, which is assembled with the element stiffness matrices. $\{u\}$ is the unknown nodal deflection vector and $\{P\}$ is the external load vector, which is commonly presumed unchangeable during the optimization. As is well known, the movement of a support would change the boundary conditions of the system and consequently, lead to the system stiffness redistribution. Differentiating Eq. (3) with respect to the support position yields

$$\frac{\mathrm{d}[K]}{\mathrm{d}a}\{u\} + [K]\frac{\mathrm{d}\{u\}}{\mathrm{d}a} = 0 \tag{4}$$

Then, by simple manipulations, we obtain

$$\frac{\mathrm{d}\{u\}}{\mathrm{d}a} = -[K]^{-1}\frac{\mathrm{d}[K]}{\mathrm{d}a}\{u\} \tag{5}$$

Premultiplying both sides of Eq. (5) by a virtual unit force vector $\{F^i\}^T$, in which, only the term corresponding to the ith deflection is equal to a unit and others are zeroes. Thus, the derivative of the ith deflection $\frac{\mathrm{d}\delta_i}{\mathrm{d}a}$ is obtained

$$\frac{\mathrm{d}\delta_i}{\mathrm{d}a} = -\{F^i\}^T[K]^{-1}\frac{\mathrm{d}[K]}{\mathrm{d}a}\{u\} = -\{u^i\}^T\frac{\mathrm{d}[K]}{\mathrm{d}a}\{u\} \tag{6}$$

where $\{u^i\}$ is the virtual nodal deflection vector caused by $\{F^i\}$, and the symmetry of the global stiffness matrix $[K]$ has been used. Once the derivative of the global stiffness matrix of the system is found, the derivative of the ith deflection can be calculated simply. Next, two kinds of structures will be studied in sequence as to the movement of a simple support.

3.2. Beam element with elastic support

Let us first consider a uniform beam element of length L_e with an elastic simple support in its span, as shown in Fig. 1. The element is modeled as a classical Euler–Bernoulli beam so that the shear deformation of the element is ignored. The in-span spring support with the stiffness k is located at a and acts only in the transverse direction. Therefore, the transverse displacement at the supported point can be approximated with the nodal transverse displacements and slopes of the element

$$\delta(a) = [N_1 \quad N_2 \quad N_3 \quad N_4]_{(a)} \cdot \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{7}$$

where N_{1-4} are the shape functions of the beam element, which should be independent of the boundary conditions. The strain energy U in the spring support can be expressed in a quadratic form as

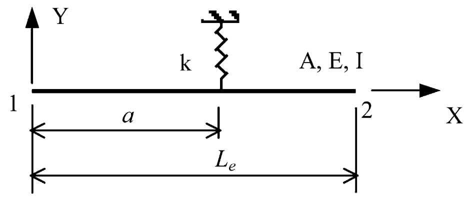

Fig. 1. Beam element with an elastic support attachment.

$$U = \frac{1}{2}k\delta^2(a) = \frac{k}{2}[\,v_1 \quad \theta_1 \quad v_2 \quad \theta_2\,]\begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ & N_2^2 & N_2N_3 & N_2N_4 \\ & & N_3^2 & N_3N_4 \\ & \text{Sym.} & & N_4^2 \end{bmatrix}_{(a)} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{8}$$

Then, the equivalent stiffness matrix of the in-span spring can be obtained

$$[K]_S = k\begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ & N_2^2 & N_2N_3 & N_2N_4 \\ & & N_3^2 & N_3N_4 \\ & \text{Sym.} & & N_4^2 \end{bmatrix}_{(a)} \tag{9}$$

As we know, the support shift would not alter the element stiffness. Then, the sensitivity of the ith nodal deflection with respect to the spring support position can be obtained according to Eq. (6)

$$\frac{\mathrm{d}\delta_i}{\mathrm{d}a} = -\{u_\mathrm{e}^i\}^T\frac{\mathrm{d}[K]_S}{\mathrm{d}a}\{u_\mathrm{e}\} \tag{10}$$

where $\{u_\mathrm{e}^i\}$ and $\{u_\mathrm{e}\}$ are the nodal deflection quantities of the related element caused by virtual and real loadings, respectively. Usually, the Hermite functions are adopted for the Euler–Bernoulli beam element (Zhu, 1998)

$$\begin{cases} N_1 = 1 - 3\left(\frac{a}{L_\mathrm{e}}\right)^2 + 2\left(\frac{a}{L_\mathrm{e}}\right)^3, \quad N_2 = a - 2L_\mathrm{e}\left(\frac{a}{L_\mathrm{e}}\right)^2 + L_\mathrm{e}\left(\frac{a}{L_\mathrm{e}}\right)^3 \\ N_3 = 3\left(\frac{a}{L_\mathrm{e}}\right)^2 - 2\left(\frac{a}{L_\mathrm{e}}\right)^3, \quad N_4 = -L_\mathrm{e}\left(\frac{a}{L_\mathrm{e}}\right)^2 + L_\mathrm{e}\left(\frac{a}{L_\mathrm{e}}\right)^3 \end{cases} \tag{11}$$

Provided that the support is attached only at one of the two ends of the beam element, hence the derivative of the equivalent stiffness matrix is achieved either

$$\left.\frac{\mathrm{d}[K]_S}{\mathrm{d}a}\right|_{a=0} = \begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ or } \left.\frac{\mathrm{d}[K]_S}{\mathrm{d}a}\right|_{a=L_\mathrm{e}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k & 0 \end{bmatrix} \tag{12}$$

Substituting the above expressions back into Eq. (10) gives the sensitivity of the ith nodal deflection, respectively,

$$\left.\frac{\mathrm{d}\delta_i}{\mathrm{d}a}\right|_{a=0} = -k(v_1^i \cdot \theta_1 + \theta_1^i \cdot v_1), \text{ or } \left.\frac{\mathrm{d}\delta_i}{\mathrm{d}a}\right|_{a=L_\mathrm{e}} = -k(v_2^i \cdot \theta_2 + \theta_2^i \cdot v_2) \tag{13}$$

where v_1 and θ_1 are, respectively, the transverse displacement and the slope at End 1 of the beam element caused by real loading, and v_2 and θ_2 are the corresponding items at End 2. The superscript i indicates that the associated entries are caused by the virtual unit force.

Since the continuity of the nodal displacement and slope is always imposed between two neighboring elements, it is recognizable that the sensitivities obtained with Eq. (13) are the same at an FE node from two adjacent elements. Therefore, the subscripts symbolically indicating the element end in Eq. (13) will be eliminated subsequently, and the nodal deflection quantities are just related to the point at which the support is positioned.

For Euler–Bernoulli beam element, we can get the following relationship between the transverse deflection and the slope (Zhu, 1998):

$$\theta = v' = \frac{\mathrm{d}v}{\mathrm{d}x} \tag{14}$$

Therefore, the sensitivity of the nodal deflection with respect to the elastic support position becomes

$$\frac{\mathrm{d}\delta_i}{\mathrm{d}a} = -k\big(v^i(a)v'(a) + v^{i\prime}(a)v(a)\big) \tag{15}$$

where $v(a)$ and $v'(a)$ represent the transverse displacement and its derivative of the beam evaluated at the supported point by the real loading, respectively. $v^i(a)$ and $v^{i\prime}(a)$ are the corresponding entries by the virtual unit force. Eq. (15) indicates that the deflection sensitivity is proportional to the support stiffness. This formula can be further simplified by introducing the spring force or support reaction force R

$$R = -kv(a) \tag{16}$$

Therefore, the design sensitivity of the nodal deflection can be evaluated with

$$\frac{\mathrm{d}\delta_i}{\mathrm{d}a} = R^i v'(a) + Rv^{i\prime}(a) \tag{17}$$

where R and R^i are the support reaction forces for real and virtual loadings, respectively. Eq. (17) shows that the design sensitivity can be calculated directly using the results available from the FE analysis under real and virtual loadings, respectively.

Once the stiffness of a spring increases to infinite, the elastic support will become a rigid one and the transverse deflection of the supported point reduces to zero. In this situation, Eq. (17) is still valid to evaluate the deflection sensitivity with regard to the movement of a rigid support without any difficulties.

After acquiring, respectively, the deflections and the associated derivatives with the support locating at the nodes of the structural FE model, it is easy to estimate the deflection value with the support within the element span by the Hermite interpolation technique:

$$\delta_i(a) = [\,N_1 \quad N_2 \quad N_3 \quad N_4\,] \cdot \begin{Bmatrix} \delta_i(0) \\ \frac{\mathrm{d}\delta_i(0)}{\mathrm{d}a} \\ \delta_i(L_\mathrm{e}) \\ \frac{\mathrm{d}\delta_i(L_\mathrm{e})}{\mathrm{d}a} \end{Bmatrix} \tag{18}$$

where N_{1-4} are determined by Eq. (11).

Much as in the practical problem, it is not always the case that the optimal support position is exactly at the FE node of the structure. Regularly, the optimal position of a support is located within the element span. In this case, Eq. (18) can be used to estimate the optimal support position as will be described in the illustrative Example 5.1.

3.3. Plate element with elastic support

In this subsection, the preceding work is extended to a thin plate element, i.e., the classical Kirchhoff flexural element with an elastic support attachment in its region, seeing Fig. 2. Similarly, the transverse displacement of the supported point along the z-axis can be expressed in terms of the element nodal displacements and slopes

$$w(a,b) = [N]_{(a,b)} \cdot \{u\}_\mathrm{e} \tag{19}$$

where $[N]$ is a row vector of shape functions of the rectangular plate element and $\{u\}_\mathrm{e}$ is a column vector of element nodal degrees of freedom

$$[N] = [\,N_1 \quad N_{x1} \quad N_{y1} \quad N_2 \quad N_{x2} \quad N_{y2} \quad N_3 \quad N_{x3} \quad N_{y3} \quad N_4 \quad N_{x4} \quad N_{y4}\,] \tag{20a}$$

$$\{u\}_\mathrm{e} = [\,w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \quad w_4 \quad \theta_{x4} \quad \theta_{y4}\,]^T \tag{20b}$$

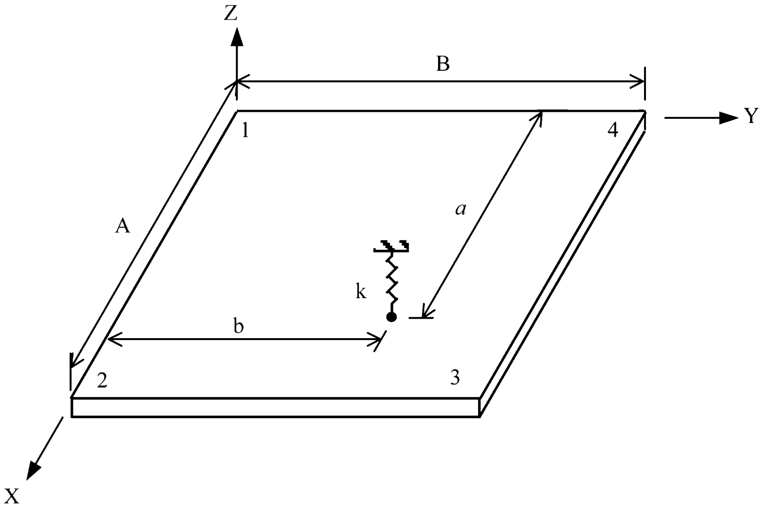

Fig. 2. Thin plate element with an elastic support attachment.

Similar to the preceding derivations, the equivalent stiffness matrix of the elastic support is given as

$$[K]_S = k\begin{bmatrix} N_1^2 & N_1N_{x1} & N_1N_{y1} & \cdots & N_1N_{x4} & N_1N_{y4} \\ & N_{x1}^2 & N_{x1}N_{y1} & \cdots & N_{x1}N_{x4} & N_{x1}N_{y4} \\ & & N_{y1}^2 & \cdots & N_{y1}N_{x4} & N_{y1}N_{y4} \\ & & & \vdots & \vdots & \vdots \\ & \text{Sym.} & & & & N_{y4}^2 \end{bmatrix}_{12\times 12} \tag{21}$$

The sensitivity of the nodal deflection is obtained according to Eq. (6)

$$\frac{\partial \delta_i}{\partial a} = -\{u_e^i\}^T \frac{\partial [K]_S}{\partial a}\{u_e\} \tag{22a}$$

$$\frac{\partial \delta_i}{\partial b} = -\{u_e^i\}^T \frac{\partial [K]_S}{\partial b}\{u_e\} \tag{22b}$$

Now let us take the standard shape functions of a rectangular thin plate element and utilize their characteristics at the element corner vertices (Zhu, 1998). Assume that the support is attached at Vertex 1 of the element, seeing Fig. 2, then one gets:

$$N_1 = \frac{\partial N_{x1}}{\partial y} = -\frac{\partial N_{y1}}{\partial x} = 1 \tag{23}$$

$$N_{x1} = N_{y1} = \frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial y} = \frac{\partial N_{x1}}{\partial x} = \frac{\partial N_{y1}}{\partial y} = 0 \tag{24}$$

Other shape functions and their first derivatives are all nulls. After manipulations similar to those before, the sensitivity of the ith deflection is given as

$$\left.\frac{\partial \delta_i}{\partial a}\right|_{a=0,\, b=0} = k\left(w_1^i\theta_{y1} + \theta_{y1}^i w_1\right) \tag{25a}$$

$$\left.\frac{\partial \delta_i}{\partial b}\right|_{a=0,\, b=0} = -k\left(w_1^i \theta_{x1} + \theta_{x1}^i w_1\right) \tag{25b}$$

where w_1, θ_{x1} and θ_{y1} are, respectively, the actual transverse displacement and slopes along the x and y axes at Vertex 1 of the plate element. w_1^i, θ_{x1}^i and θ_{y1}^i denote the associated virtual entries.

Likewise, the design sensitivities of the support at other corner vertices of the element can be conducted. Due to the compatibility of the nodal degrees of freedom between adjacent elements (though there exist discontinuities for the normal slope across the common edges of adjacent elements), the resulting formulas are the same as Eq. (25). Therefore, the subscripts indicating the element vertex in Eq. (25) will be suppressed subsequently.

From the classical Kirchhoff flexural theory, there exist the following relations between the transverse displacements and slopes

$$\theta_x = w_{,y} = \frac{\partial w}{\partial y}, \quad \theta_y = -w_{,x} = -\frac{\partial w}{\partial x} \tag{26}$$

Moreover, the support reaction force R can be calculated with

$$R = -kw(a, b) \tag{27}$$

By substituting Eqs. (26) and (27) into Eq. (25), the deflection sensitivity with respect to a support position can be obtained as:

$$\frac{\partial \delta_i}{\partial a} = R^i w_{,x}(a, b) + w_{,x}^i(a, b) R \tag{28a}$$

$$\frac{\partial \delta_i}{\partial b} = R^i w_{,y}(a, b) + w_{,y}^i(a, b) R \tag{28b}$$

where $w_{,x}(a, b)$ and $w_{,y}(a, b)$ indicate, respectively, the partial derivatives of $w(a, b)$ with respect to the related axes of the Cartesian system, evaluated at the support point (a, b) under the real loading case. $w_{,x}^i(a, b)$ and $w_{,y}^i(a, b)$ correspond to the entries under virtual unit force. Similarly, as the elastic support becomes a rigid one, Eq. (28) is still valid.

So far, it has been implied that a support shifts along the edges of the element. If the elementary edges are not parallel to the global axes or the support moves along a specified direction, then the directional derivative can be calculated by taking advantage of the gradient of the deflection

$$\frac{\mathrm{d}\delta_i}{\mathrm{d}\mathbf{s}} = \mathrm{grad}(\delta_i) \cdot \mathrm{d}\mathbf{s} = \frac{\partial \delta_i}{\partial a} \cos\alpha + \frac{\partial \delta_i}{\partial b} \cos\beta \tag{29}$$

where $\{\cos\alpha, \cos\beta\}$ are the directional cosines of the specified direction in the global Cartesian system.

4. Optimization procedure for support positions

After acquiring the deflection sensitivity, it is able to apply the results to support position optimization of beam or plate structures. In this paper, a heuristic optimization algorithm, based on the ideas of the ESO method (Xie and Steven, 1997), is employed to optimize the support positions gradually for minimizing the maximal absolute deflection of a structure. This heuristic optimization algorithm, which is better suitable for discrete design variable problems, consists of two steps. The first step is to find the most efficient support on the basis of design sensitivity analysis. The second step is to shift the support to the nearby node so as to decrease the maximal absolute deflection. Supports are assumed to move along the elementary edges with the interval of elementary size. The search direction of a support is determined according to the design sensitivity. Once oscillation of the maximal absolute deflection occurs between two nodes with support movements, the interpolation technique is used to estimate the optimal position within the element, since in this case, the minimum of the maximal absolute deflection may take place with the support locating in the element span.

Usually, the deflection change can be written linearly in terms of the position variation:

$$\Delta\delta_i \approx \frac{\partial\delta_i}{\partial a_j} \cdot \Delta a_j \tag{30}$$

In order to reduce a specified deflection δ_i (>0) greater with less modification of the support positions, it is desirable to shift the support with the maximal absolute value of the design sensitivity among all the possibly movable supports

$$\mathrm{Max}\left\{ \left| \frac{\partial\delta_i}{\partial a_j} \right|, \quad j = 1, \ldots, n \right\} \tag{31}$$

and the move direction of the support is determined by

$$\mathrm{sign}(\Delta a_j) = -\mathrm{sign}\left(\frac{\partial\delta_i}{\partial a_j} \right) \tag{32}$$

where Δa_j is the move interval of the jth support position and $\mathrm{sign}(\cdot)$ is the sign function.

To implement the optimization procedure, the FE method is used to calculate the structural deflections numerically under real and virtual loadings. Then, the associated displacements (or support reaction forces) and slopes at the supported points are accessible and the design sensitivities can be computed immediately.

5. Illustrative examples

To demonstrate the validity of the developed deflection sensitivity and the proposed optimization algorithm, three numerical examples are employed to illustrate support position optimization for minimizing the maximal absolute deflection of a structure. Despite their simplicities, no theoretical solutions exist for their position optimizations. The program is developed based on the commercial software SAMCEF®/Asef.

5.1. Simply supported beam

This example is devised to illustrate the deflection sensitivities and to visualize the optimization solutions. A uniform beam of length $L = 2$ m is simply supported with two rigid supports. The cross-section is a square with its side $H = 0.1$ m. The beam is discretized evenly with 10 elements as shown in Fig. 3. Let Young's modulus $E = 2.1 \times 10^{11}$ Pa and material density $\rho = 7800$ kg/m^3. A concentrated load $P_c = 200$ kN acts at the mid-span of the beam while two concentrated loads $P_E = 100$ kN act at its two ends, respectively. Additionally, the structural weight is also under consideration with the gravity acceleration $g = -9.81$ m/s^2. The two supports needs to be relocated symmetrically to minimize the maximal absolute deflection of the beam. As to this simple structure, it can be anticipated that the maximal absolute deflection would switch its position between the beam's ends and its center with the movement of the support position. Then, deflections only at those points are monitored. In order to testify the proposed algorithm, the optimization process starts from two different initial designs with the supports at $X/L = 1.0$ and 0.0, respectively.

Fig. 4 shows the optimization histories of the beam deflections at both its end and mid-span with the rigid support movement. The deflection shapes of the beam with different support positions are plotted in

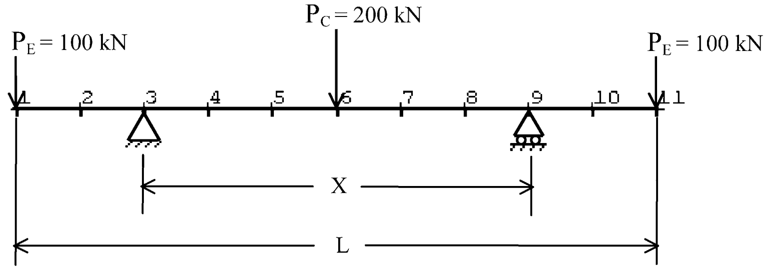

Fig. 3. Simply supported beam and its FE mesh.

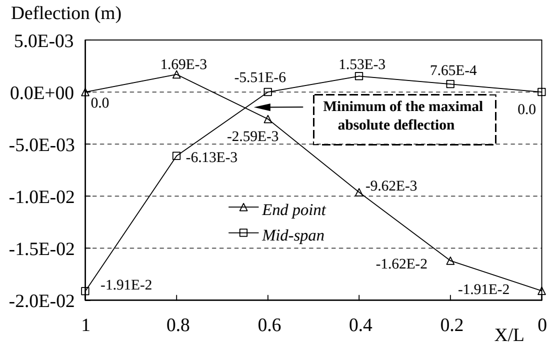

Fig. 4. Evolutionary histories of beam's deflections.

Fig. 5. Table 1 lists the displacements, design sensitivities and the maximal absolute displacement of the beam with different support positions. It is seen clearly that when the simple supports hold up the beam at its ends, the mid-span of the beam deflects maximally. Conversely, when the supports hold up the beam at its mid-span, both beam's ends deflect maximally. The structural gravity brings about the maximal deflections unequal in those two cases. As the support moves from $X/L = 0.8$ to 0.6, i.e., from Node 2 (10) to Node 3 (9), the maximal absolute deflection switches its position from the beam mid-span to its ends. At

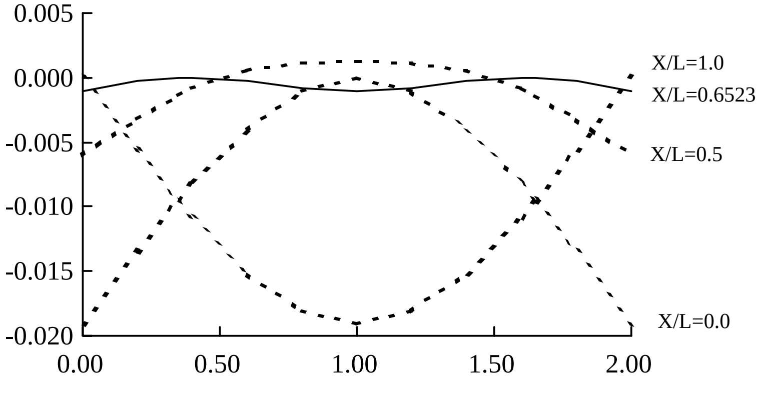

Fig. 5. Deflection shapes of the beam for different support positions.

Table 1
Beam deflections (m) and design sensitivities with different support positions

Support position X/L	Beam deflections δ ($\times 10^{-3}$)			Support sensitivities ($\times 10^{-3}$)	
	At end	At mid-span	Max$\lvert\delta\rvert$	At end	At mid-span
1.0	0.0	−19.139	19.139	28.718	86.082[a]
0.8	1.690	−6.130	6.130	−9.146	45.924[a]
0.6	−2.588	−0.006	2.588	−30.953[a]	17.233
0.4	−9.621	1.528	9.621	−36.700[a]	0.013
0.2	−16.197	0.765	16.197	−26.384[a]	−5.733
0.0	−19.103	0.0	19.103		

[a] Sensitivity of the maximal deflection.

Table 2
Maximal absolute deflections of the beam with different support stiffnesses

Support stiffness	Maximal absolute deflection (mm)
100 EI/L^3 (2.1875×10^7 N/m)	10.249
200 EI/L^3 (4.375×10^7 N/m)	5.661
Rigid	1.072

the same time, the sensitivity of the maximal absolute displacement alters its sign, seeing Table 1. With the relevant displacements and the sensitivities, one can predict the switching point a^* by imposing the equality condition:

$$\delta_1(a^*) = \delta_6(a^*)$$

That is, with the Hermite interpolation technique within the element, one can get the following equation

$$[N_1 \quad N_2 \quad N_3 \quad N_4]_{(a^*)} \begin{Bmatrix} \delta_1|_{\text{Node 2}} \\ \frac{d\delta_1}{da}\Big|_{\text{Node 2}} \\ \delta_1|_{\text{Node 3}} \\ \frac{d\delta_1}{da}\Big|_{\text{Node 3}} \end{Bmatrix} = [N_1 \quad N_2 \quad N_3 \quad N_4]_{(a^*)} \begin{Bmatrix} \delta_6|_{\text{Node 2}} \\ \frac{d\delta_6}{da}\Big|_{\text{Node 2}} \\ \delta_6|_{\text{Node 3}} \\ \frac{d\delta_6}{da}\Big|_{\text{Node 3}} \end{Bmatrix}$$

This is a cubic equation about a^* and can be solved simply with Cardano's formulae. At last, the optimal support position is achieved with $X/L = 0.6523$. The minimum of the maximal absolute deflection is 1.072 mm, only 5.6% of that with the supports locating at the beam's ends. Substantial reduction of the maximal deflection has been achieved with the optimal design of support positions.

Furthermore, it is found that the resulting solution is also valid for elastic supports with different stiffnesses in this numerical example. This is because the reaction forces of the supports and the flexural deformation of the beam are the same for all cases of supports of different stiffnesses. So are the design sensitivities. The minimum values of the maximal absolute deflections of the beam with different support stiffnesses are listed in Table 2 for comparison. The discrepancies are, in fact, produced by the deformations of the elastic supports themselves.

5.2. Frame structure

A frame structure, loaded by four concentrated forces together with its self-weight, is shown in Fig. 6. The original deflection of the frame without the lateral supports attachment takes the maximal value, $|\delta_{\max}| = 211.63$ mm, at its free tip. Now a pair of rigid supports is attached laterally at the nodes on the lower chord to reduce its maximal deflection. The interval of the two simple supports remains 1 m apart fixedly. Hence, a is the exclusive independent coordinate of the supports.

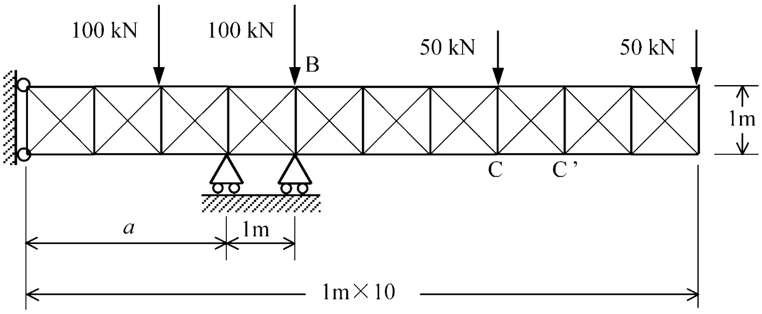

Fig. 6. Frame structure with the applied loads and the additional lateral supports.

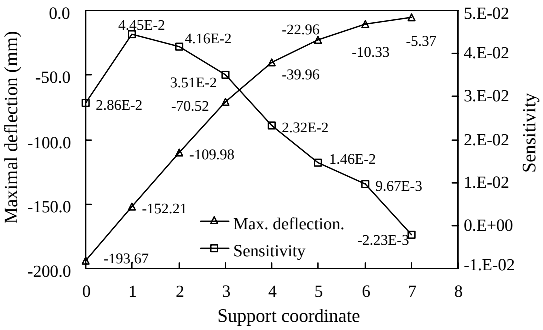

Fig. 7. Variations of the maximal deflection and design sensitivity of the frame versus the support position.

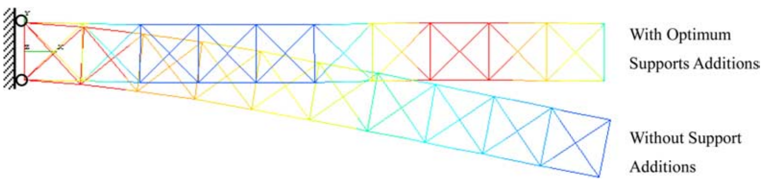

Fig. 8. Comparison of the deformation shapes of the frame structure.

The members of the frame structure are separated into two groups. The cross section of all the diagonal members is full circle with diameter $D = 20$ mm. Other members are of tubular cross section with the outer diameter $D_o = 80$ mm and inner diameter $D_i = 60$ mm. Young's modulus is $E = 2.1 \times 10^{11}$ Pa and material density $\rho = 7800$ kg/m^3. The optimization process starts from the initial design at $a = 0$. Deflections of all nodes on the upper chord are under control.

The variations of the maximal deflection of the frame and the design sensitivity are shown in Fig. 7, respectively. The sensitivity is positive until $a = 6$ m. Afterwards, when $a = 7$ m, i.e., the supports hold the frame at C and C', the sensitivity becomes negative. Meanwhile, the maximal absolute deflection, occurring at point B, reaches minimum $|\delta_{max}| = 5.37$ mm, only 2.54% of that without the additional supports. Fig. 8 shows the final deformation of the frame compared with the original. Apparently, the frame deforms more evenly with a pair of the supports locating at the optimal position.

5.3. Simply supported rectangular plate

A rectangular plate, shown in Fig. 9, is loaded by a concentrated force of 2 kN at its central point C and a uniformly distributed force of 2 kN/m^2 along the transverse direction. The plate is symmetrically supported with four elastic supports on its diagonals. The thickness of the plate is 1 cm uniformly. The plate is discretized regularly with a mesh of 10×10 quadrilateral elements. Young's modulus is $E = 73.1$ GPa and Poisson ratio $v = 0.3$. The four elastic supports, with stiffness 10^6 N/m, will move along the diagonals of the plate to minimize the plate deflection. The optimization process starts from the initial design at $a/L = 0$.

Fig. 10 shows the variation of the maximal deflection of the plate versus the support positions. The numbers in the parentheses are the design sensitivities of the maximal deflection. When supports hold the

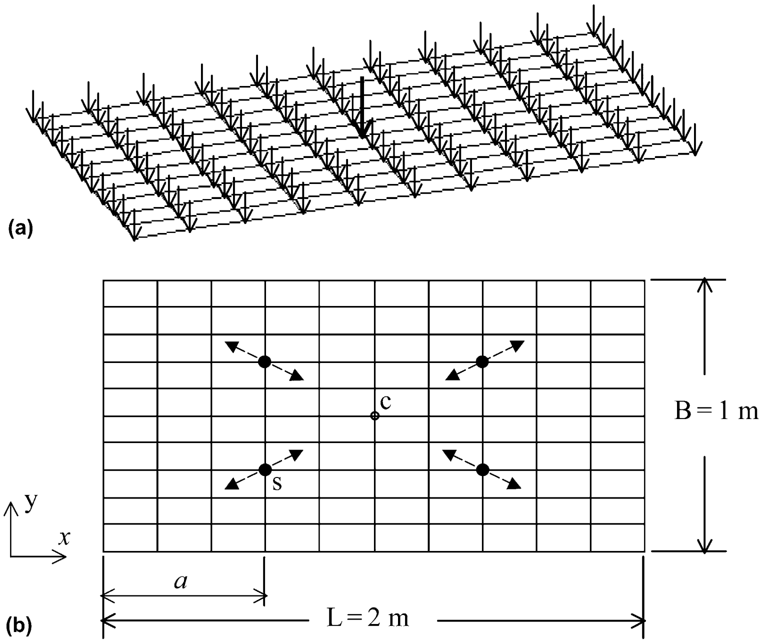

Fig. 9. Rectangular plate: (a) with applied forces; (b) with four simple supports.

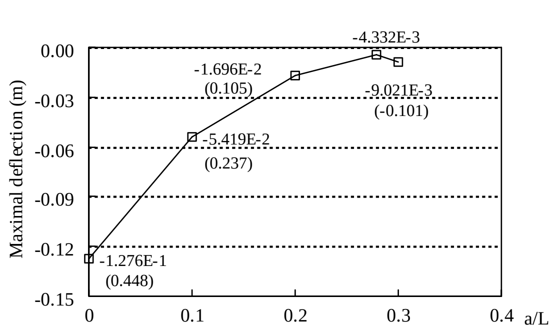

Fig. 10. Variation of the maximal deflection of the rectangular plate versus the support position.

plate at its corners, i.e., $a/L = 0$, the maximal absolute deflection is 0.1276 m, occurring at the mid-span of the long edge. When the supports move from $a/L = 0.2$ to 0.3, the design sensitivity changes its value from 0.105 to −0.101. Thus, it is known that the optimal position is in this span and can be interpolated with the obtained results. At the optimal design of the support positions $a/L = 0.2791$, the maximal absolute deflection is 4.332×10^{-3} m, only 3.4% of the initial value, occurring at the corners and the center of the plate. Obviously, the optimal design of support positions has reduced the plate deflection significantly.

6. Conclusions

Structural optimization often requires the evaluation of design sensitivities. In this paper, the closed-form expressions for the deflection sensitivity of a beam or plate structure with respect to a support position are first developed by the discrete method. Both elastic and rigid supports are taken into account. Furthermore, based on the sensitivity analysis, a heuristic optimization algorithm, called the evolutionary shift method, is presented for support position optimization with the objective of minimizing the maximal deflection. To facilitate the convergence of the process, a polynomial interpolation technique is used to evaluate the solution more accurately. Three numerical examples are utilized to demonstrate the validity of the sensitivity formulation and the effectiveness of the proposed optimization method.

In practical problems, support positions are of great importance to provide additional rigidity and improve the structural properties. Results of the examples show that the optimal design of support positions can reduce the structural deflection substantially without additional material.

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